FYS3415 / FYS4415

Advanced Quantum Mechanics

Midterm Examination – Solutions

Problem 1: Infinite Square Well

Inside the well ($0 < x < L$) the Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

This gives

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad k^2 = \frac{2mE}{\hbar^2}.$$

(a) Eigenfunctions

General solution:

$$\psi(x) = A\sin(kx) + B\cos(kx).$$

Boundary conditions:

$$\psi(0) = 0 \Rightarrow B = 0,$$

$$\psi(L) = 0 \Rightarrow \sin(kL) = 0 \Rightarrow kL = n\pi.$$

Thus

$$k_n = \frac{n\pi}{L}.$$

Normalization:

$$\int_0^L |\psi_n|^2 dx = 1.$$

$$A^2\frac{L}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{L}}.$$

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)}$$

(b) Energy levels

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

(c) Expectation value $\langle x \rangle$

$$\langle x \rangle = \int_0^L \psi_n^*(x) x \psi_n(x) dx.$$

Using symmetry of $\sin^2$ about $L/2$:

$$\boxed{\langle x \rangle = \frac{L}{2}}$$

(d) Expectation value $\langle p \rangle$

$$\hat{p} = -i\hbar \frac{d}{dx}.$$

Since ψ_n is real and odd/even symmetric within the well:

$$\boxed{\langle p \rangle = 0}$$

Problem 2: Harmonic Oscillator

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).$$

(a) Energy spectrum

Define number operator:

$$\hat{N} = a^\dagger a.$$

Using $[a, a^\dagger] = 1$ one finds

$$\hat{N}|n\rangle = n|n\rangle.$$

Thus

$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle.$$

$$\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$$

(b) Expectation value $\langle x^2 \rangle$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger).$$

$$x^2 = \frac{\hbar}{2m\omega}(a + a^\dagger)^2.$$

Evaluating in state $|n\rangle$:

$$\boxed{\langle x^2 \rangle_n = \frac{\hbar}{2m\omega}(2n + 1)}$$

(c) Ground state wavefunction

From $a|0\rangle = 0$:

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)}$$

Problem 3: Uncertainty Principle

(a) General relation

Using Cauchy–Schwarz inequality:

$$\sigma_A^2 \sigma_B^2 \geq \left| \frac{1}{2} \langle [A, B] \rangle \right|^2 .$$

$$\boxed{\sigma_A \sigma_B \geq \frac{1}{2} |\langle [A, B] \rangle|}$$

(b) Position and momentum

$$[x, p] = i\hbar.$$

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

(c) Saturation

Equality holds for states satisfying

$$(A - \langle A \rangle)\psi = i\lambda(B - \langle B \rangle)\psi.$$

For x and p, these are Gaussian wave packets (coherent states).

Problem 4: Two-Level System

$$H = \begin{pmatrix} E_0 & \Delta \\ \Delta & E_0 \end{pmatrix}.$$

(a) Eigenvalues and eigenvectors

$$\det(H - \lambda I) = 0 \Rightarrow (E_0 - \lambda)^2 - \Delta^2 = 0.$$

$$\boxed{E_\pm = E_0 \pm \Delta}$$

Normalized eigenvectors:

$$\boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}$$

(b) Time evolution

Initial state:

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle).$$

Time evolution:

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar} \begin{pmatrix} \cos(\Delta t/\hbar) \\ -i \sin(\Delta t/\hbar) \end{pmatrix}.$$

(c) Transition probability

$$\boxed{P_{1\to 2}(t) = \sin^2 \left(\frac{\Delta t}{\hbar} \right)}$$

Rabi frequency:

$$\Omega = \frac{2\Delta}{\hbar}.$$